

# Construction and separability of nonlinear soliton integrable couplings

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## Abstract

A very natural construction of integrable extensions of soliton systems is presented. The extension is made on the level of evolution equations by a modification of the algebra of dynamical fields. The paper is motivated by recent works of Wen-Xiu Ma et al. (Comp. Math. Appl. **60** (2010) 2601, Appl. Math. Comp. **217** (2011) 7238), where new class of soliton systems, being nonlinear integrable couplings, was introduced. The general form of solutions of the considered class of coupled systems is described. Moreover, the decoupling procedure is derived, which is also applicable to several other coupling systems from the literature.

## 1 Introduction

For a given nonlinear integrable dynamical system there usually exists many different integrable extensions which equations of motion take a triangular form. They are usually called triangular systems. For Liouville integrable nonlinear ODE's the interesting class of triangular systems is given in [1]. For soliton systems we know more examples. Triangular extensions of the KdV system was considered in [2]. In [3] there are constructed linear extensions of soliton systems, also taking the triangular form, being by the author called 'dark equations'. Another class of linear extensions the so-called linear integrable couplings of soliton systems was introduced in [4] and then developed in [5] and many other papers. Recently, the theory of nonlinear integrable couplings of ordinary soliton systems was presented in [6] and [7].

In the present paper we introduce very natural triangular extensions of basic integrable nonlinear systems, which from the construction are also integrable. Moreover their solutions are uniquely determined by the solutions of the basic equation. This extensions are made on the level of evolution equations by a modification of the algebra of dynamical fields. The construction is motivated by nonlinear integrable couplings recently introduced in [6, 7]. We also propose the decoupling procedure for the considered class of integrable couplings including those from [6, 7] and [8, 9, 10].

In Section 2 we introduce the algebra of coupled scalars, which is the underlying algebra for the nonlinear integrable extensions defined in Section 3. We derive soliton integrable couplings,

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both of field and lattice type. In Section 4, the general form of solution of the coupled systems is obtained. An example of soliton solutions of the nonlinearly coupled KdV system is presented. In Section 5 we derive a matrix representation of the algebra of coupled scalars and in consequence the matrix Lax representations for nonlinear couplings constructed in previous sections. Finally, in Section 6, we prove that any member of the constructed family of coupled systems separates into copies of the original soliton system. We also show the source of the decoupling procedure.

## 2 The algebra of coupled scalars

Consider  $n$ -dimensional vector space over field of real numbers  $\mathbb{R}$ . We define an algebra structure by the following multiplication

$$\mathbf{e}_i \cdot \mathbf{e}_j := \mathbf{e}_{\max(i,j)}, \quad (1)$$

where  $\mathbf{e}_i$  are the basis vectors. Let  $\mathbf{a} = \sum_{i=1}^n a_i \mathbf{e}_i$ , then

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{c},$$

where

$$c_i = a_i b_i + a_i \left( \sum_{k=1}^{i-1} b_k \right) + \left( \sum_{k=1}^{i-1} a_k \right) b_i.$$

We find that the value of the coefficient  $c_i$  is given by  $a_i b_i$  plus terms depending on lower order elements,  $a_k, b_k$  with  $k < i$ . Therefore, we call this algebra as an algebra of coupled scalars. This algebra is unital, commutative and associative, and  $\mathbf{e}_i$  are idempotent elements, what follows immediately from the definition (1). The unity element is  $\mathbf{e}_1$ .

For  $n = 4$  we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 + a_2 b_1 + a_1 b_2 \\ a_3 b_3 + a_3(b_1 + b_2) + (a_1 + a_2)b_3 \\ a_4 b_4 + a_4(b_1 + b_2 + b_3) + (a_1 + a_2 + a_3)b_4 \end{pmatrix}.$$

In the algebra of coupled scalars, for product of  $m$  elements the following formula holds:

$$\mathbf{a}^1 \cdot \dots \cdot \mathbf{a}^m = \begin{pmatrix} a_1^1 \\ \vdots \\ a_n^1 \end{pmatrix} \cdot \dots \cdot \begin{pmatrix} a_1^m \\ \vdots \\ a_n^m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \mathbf{b}, \quad (2a)$$

where

$$b_k = \prod_{j=1}^m \left( \sum_{r=1}^k a_r^j \right) - \prod_{j=1}^m \left( \sum_{r=1}^{k-1} a_r^j \right). \quad (2b)$$

## 3 Nonlinear couplings of soliton systems

Consider a commutative and associative algebra, with respect to the ordinary dot multiplication, of smooth functions on  $\mathbb{R}^m$  with  $m$  derivations  $\frac{\partial}{\partial x_i} : C^\infty(\mathbb{R}^m) \rightarrow C^\infty(\mathbb{R}^m)$ . Let us construct its

coupled counterpart  $C_d^\infty(\mathbb{R}^m)$ , that is an algebra of functions

$$\mathbf{f}(x) = f_1(x)\mathbf{e}_1 + \dots + f_n(x)\mathbf{e}_n = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix},$$

where  $x = (x_1, \dots, x_m)$ , taking values in the algebra of coupled scalars. So, it is commutative and associative algebra with respect to the multiplication (1) and the derivations in  $C_d^\infty(\mathbb{R}^m)$  can be defines by

$$\frac{\partial}{\partial x_i} := \frac{\partial}{\partial x_i} \mathbf{e}_1 \quad i = 1, \dots, m.$$

These derivations are well-defined as

$$\frac{\partial}{\partial x_i} \mathbf{e}_1 \cdot (f \cdot g) = \left( \frac{\partial}{\partial x_i} \mathbf{e}_1 \cdot f \right) \cdot g + f \cdot \left( \frac{\partial}{\partial x_i} \mathbf{e}_1 \cdot g \right)$$

and

$$\mathbf{f}_{x_i} := \frac{\partial}{\partial x_i} \mathbf{e}_1 \cdot \mathbf{f} = \begin{pmatrix} (f_1)_{x_i} \\ \vdots \\ (f_n)_{x_i} \end{pmatrix}.$$

Here we will concentrate on the coupled extension of nonlinear integrable PDE's in  $(1+1)$ -dimension in the evolutionary form. Consider an one-field soliton system

$$u_t = K[u] \equiv K(u, u_x, u_{xx}, \dots). \quad (3)$$

Its extension to the system of coupled PDE's takes the form

$$\mathbf{u}_t = K[\mathbf{u}] \equiv K(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots), \quad (4)$$

where

$$\mathbf{u} = u_1 \mathbf{e}_1 + \dots + u_n \mathbf{e}_n = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \quad u_1 = u,$$

and in (4) the ordinary dot multiplication is replaced by (1).

**Lemma 1** *The system of coupled equations (4) takes the form*

$$\begin{aligned} (u_1)_t &= K[u_1] \\ (u_k)_t &= K \left[ \sum_{i=1}^k u_i \right] - K \left[ \sum_{i=1}^{k-1} u_i \right] \quad k = 2, \dots, n, \end{aligned} \quad (5)$$

where  $K[\cdot]$  is the same as in (3).

The proof follows from the power series expansion of  $K[u]$  and the relation (2). In fact, it is sufficient to consider homogenous terms as

$$(\mathbf{u}_{i_1 x} \cdot \mathbf{u}_{i_2 x} \cdot \dots \cdot \mathbf{u}_{i_m x})_k = \prod_{s=1}^m \left( \sum_{r=1}^k u_r \right)_{i_s x} - \prod_{s=1}^m \left( \sum_{r=1}^{k-1} u_r \right)_{i_s x} \quad m > 0,$$

where  $(\cdot)_k$  means  $k$ -th coefficient of  $\mathbf{u}_{i_1 x} \cdot \dots \cdot \mathbf{u}_{i_m x}$  and  $i_s$  are arbitrary nonnegative integers.

Note that the equation (5) reconstructs for  $u_1$  the basic equation. Moreover, for linear PDE's the procedure is trivial as all related coupled systems are just copies of the basic equation. The extension to the multi-field systems is obvious.

As an instructive example let us consider the  $n$ -coupled KdV ( $nc$ -KdV):

$$\begin{aligned}
\mathbf{u}_t &= \mathbf{u}_{xxx} + 6\mathbf{u} \cdot \mathbf{u}_x \\
&\Updownarrow \\
(u_1)_t &= (u_1)_{xxx} + 6u_1(u_1)_x \\
(u_2)_t &= (u_2)_{xxx} + 6u_2(u_2)_x + 6(u_1 u_2)_x \\
&\vdots \\
(u_n)_t &= (u_n)_{xxx} + 6u_n(u_n)_x + 6 \sum_{k=1}^{n-1} (u_k u_n)_x.
\end{aligned} \tag{6}$$

It is a particular case of the triangular systems. Obviously in dispersionless limit we get  $n$ -coupled dispersionless KdV ( $nc$ -dKdV).

The presented construction is also valid for differential-difference systems. If, for instance, instead of the derivative  $\frac{\partial}{\partial x_i} \mathbf{e}_1$  one defines on  $C_d^\infty(\mathbb{R}^m)$  a shift operator  $T_k \mathbf{e}_1 := \left( \exp \frac{\partial}{\partial x_k} \right) \mathbf{e}_1$ , then

$$T_k \mathbf{e}_1 \cdot \mathbf{f} = \begin{pmatrix} f_1(x_1, \dots, x_k + 1, \dots, x_m) \\ \vdots \\ f_n(x_1, \dots, x_k + 1, \dots, x_m) \end{pmatrix}.$$

As an example we consider  $nc$ -Volterra differential-difference equation:

$$\begin{aligned}
\mathbf{v}(x)_t &= \mathbf{v}(x) \cdot [\mathbf{v}(x+1) - \mathbf{v}(x-1)] \\
&\Updownarrow \\
v_1(x)_t &= v_1(x)[v_1(x+1) - v_1(x-1)] \\
v_2(x)_t &= v_2(x)[v_2(x+1) - v_2(x-1)] + v_2(x)[v_1(x+1) - v_1(x-1)] \\
&\quad + v_1(x)[v_2(x+1) - v_2(x-1)] \\
&\vdots \\
v_n(x)_t &= v_n(x)[v_n(x+1) - v_n(x-1)] + v_n(x) \left[ \sum_{i=1}^{n-1} v_i(x+1) - \sum_{i=1}^{n-1} v_i(x-1) \right] \\
&\quad + \left( \sum_{i=1}^{n-1} v_i(x) \right) [v_n(x+1) - v_n(x-1)].
\end{aligned} \tag{7}$$

The two field case, that is 2c-Volterra, is equivalent to the lattice integrable coupling system from [6].

Each coupled soliton system have a Lax and zero-curvature representation, that can be obtained by replacing an underlying algebra by its tensor product with the algebra of coupled scalars. For the above examples,  $\mathbf{L}_t = [\mathbf{L}, \mathbf{A}]$ , where the right-hand side is the usual commutator, are equations in the algebras of pseudo-differential and shift operators, respectively, over algebra of coupled functions. Actually, for the  $nc$ -KdV (6) the Lax pair is given by

$$\mathbf{L} = \partial_x^2 \mathbf{e}_1 + \mathbf{u}, \quad \mathbf{A} = 4\partial_x^3 \mathbf{e}_1 + 6\mathbf{u} \cdot \partial_x \mathbf{e}_1 + 3\mathbf{u}_x \tag{8}$$

and for  $nc$ -Volterra (7) by

$$\mathbf{L} = T\mathbf{e}_1 + \mathbf{v}(x) \cdot T^{-1}\mathbf{e}_1, \quad \mathbf{A} = T^2\mathbf{e}_1 + \mathbf{v}(x+1) + \mathbf{v}(x). \quad (9)$$

As an example of coupled extension of two-field system let us consider 2-coupled AKNS (2c-AKNS),  $n = 2$ :

$$\begin{aligned} \mathbf{p}_t &= -\frac{1}{2}\mathbf{p}_{xx} + \mathbf{p} \cdot \mathbf{p} \cdot \mathbf{q}, & \mathbf{q}_t &= \frac{1}{2}\mathbf{q}_{xx} - \mathbf{p} \cdot \mathbf{q} \cdot \mathbf{q} \\ &\Downarrow \\ p_t &= -\frac{1}{2}p_{xx} + p^2q, & q_t &= \frac{1}{2}q_{xx} - pq^2 \\ r_t &= -\frac{1}{2}r_{xx} + p^2s + 2pqr + 2prs + qr^2 + r^2s \\ s_t &= \frac{1}{2}s_{xx} - ps^2 - 2pqs - 2qrs - q^2r - rs^2, \end{aligned} \quad (10)$$

where

$$\mathbf{p} = \begin{pmatrix} p \\ r \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} q \\ s \end{pmatrix}.$$

In fact the system (10) was obtained in [8] in a different way and is equivalent to the one from [7]. Its zero-curvature representation will be presented in Section 5.

## 4 Solutions of coupling systems

The solutions of the coupled systems (4) are completely determined by solutions of the basic equations.

**Theorem 2** *Assume that  $S^1, \dots, S^n$  are arbitrary different solutions of the basic equation  $u_t = K[u]$ . Then, the solution of coupled system (4) is given by*

$$u_1 = S^1, \quad u_k = S^k - S^{k-1}, \quad k = 2, \dots, n. \quad (11)$$

Moreover, any solution of the coupled system is of the form (11).

The first part of the theorem is evident, after plugging (11) to (5) we get

$$(S^k - S^{k-1})_t - (K[S^k] - K[S^{k-1}]) = 0.$$

The proof of the second part is by induction. If the solution of the basic equation is  $u_1 = S^1$ , then the first coupled equation is

$$(u_2)_t = K[u_2 + S^1] - K[S^1] \iff (u_2 + S^1)_t = K[u_2 + S^1],$$

thus  $u_2 + S^1 = S^2$ , where  $S^2$  is another solution of the basic equation. The rest of the induction is obvious. The consequence of the theorem is that for coupled PDE's any  $n$  different solutions of the basic equation build up an appropriate solution of the coupled system.

Let us illustrate the whole construction on the example of solitons of  $nc$ -KdV (6) in the Hirota form. Let us start from 1-soliton solution. Consider  $n$  one-soliton solutions of the KdV with the same wave velocities and different phases  $\gamma_j$ :

$$S^j = 2\partial_x^2 \log(1 + f^j), \quad f^j = \exp(kx + k^3t + \gamma_j), \quad k, \gamma_j = \text{const}, \quad j = 1, \dots, n.$$

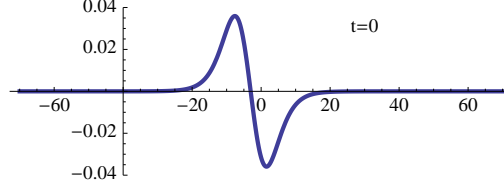


Figure 1: Coupled 1-soliton,  $u_i = S^i - S^{i-1}$  ( $i \neq 1$ ).

Then, 1-soliton solution of (6) is given in the form

$$u_1 = S^1 = 2\partial_x^2 \log(1 + f^1), \quad u_j = S^j - S^{j-1} = 2\partial_x^2 \log\left(\frac{1 + f^j}{1 + f^{j-1}}\right), \quad j = 2, \dots, n.$$

Obviously,  $u_1$  is the ordinary KdV soliton, while remaining  $u_i$  are coupled solitons (illustrated in Fig. 1), that differ among themselves by phase arguments.

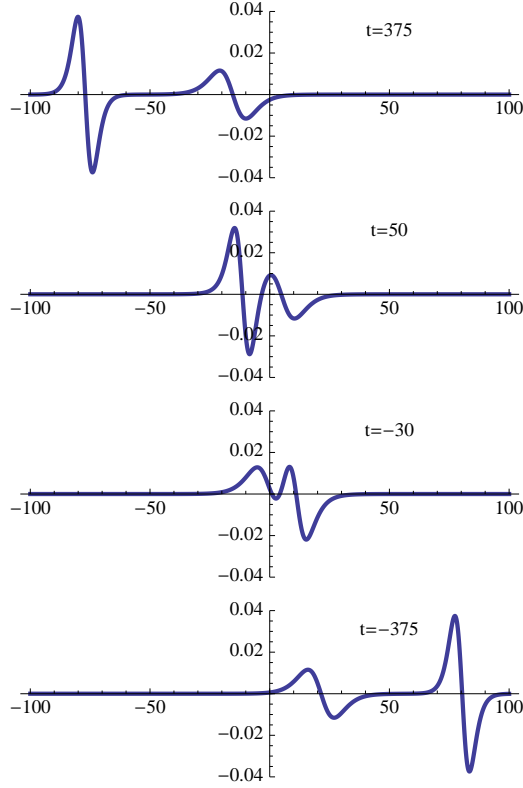


Figure 2: Coupled 2-soliton,  $u_i = S^i - S^{i-1}$  ( $i \neq 1$ ).

The coupled 2-soliton solution of  $nc$ -KdV (6) is of the form

$$u_1 = S^1 = 2\partial_x^2 \log(1 + f_1^1 + f_2^1 + af_1^1 f_2^1),$$

$$u_j = S^j - S^{j-1} = 2\partial_x^2 \log\left(\frac{1 + f_1^j + f_2^j + af_1^j f_2^j}{1 + f_1^{j-1} + f_2^{j-1} + af_1^{j-1} f_2^{j-1}}\right), \quad j = 2, \dots, n,$$

where

$$f_1^j = \exp(k_1 x + k_1^3 t + \gamma_{1j}), \quad f_2^j = \exp(k_2 x + k_2^3 t + \gamma_{2j}), \quad j = 1, \dots, n$$

and  $a = (k_1 - k_2)^2 / (k_1 + k_2)^2$ . Notice that  $u_1$  is the ordinary 2-soliton solution of the KdV, while remaining  $u_i$  are coupled 2-soliton solutions, which interaction is presented in Fig. 2.

## 5 Matrix representation of the algebra of coupled scalars

Consider  $n$  quadratic matrices  $\mathbf{E}_k$  of dimension  $n \times n$ :

$$(\mathbf{E}_k)_{ij} = \begin{cases} 1 & \text{if } j = k, i \leq k \\ 1 & \text{if } j = i, i > k \\ 0 & \text{otherwise,} \end{cases}$$

where  $k = 1, \dots, n$ . The matrices  $\mathbf{E}_i$  constitute generating elements of commutative and associative sub-algebra of triangular matrices. Let us call the matrix algebra spanned by  $\mathbf{E}_k$  as a pseudo-scalar algebra of matrices,  $ps(n)$ .

For example if  $n = 4$ :

$$\begin{aligned} \mathbf{E}_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \mathbf{E}_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \mathbf{E}_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \mathbf{E}_4 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Then, a typical element for  $n = 4$  is

$$\mathbf{A} = \sum_{i=1}^n a_i \mathbf{E}_i = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 + a_2 & a_3 & a_4 \\ 0 & 0 & a_1 + a_2 + a_3 & a_4 \\ 0 & 0 & 0 & a_1 + a_2 + a_3 + a_4 \end{pmatrix}. \quad (12)$$

**Lemma 3** *The algebra  $ps(n)$  is a matrix representation of the algebra of coupled scalars defined by the multiplication (1).*

The lemma follows immediately by showing that

$$\mathbf{E}_i \mathbf{E}_j = \mathbf{E}_j \mathbf{E}_i = \mathbf{E}_{\max(i,j)}$$

for  $i, j = 1, \dots, n$ .

The Lax representation  $\mathbf{L}_t = [\mathbf{L}, \mathbf{A}]$  of  $nc$ -KdV (8) and  $nc$ -Volterra (9) in the matrix algebra  $ps(n)$  are respectively given by

$$\mathbf{L} = \partial_x^2 \mathbf{E}_1 + U, \quad \mathbf{A} = 4\partial_x^3 \mathbf{E}_1 + 6U\partial_x \mathbf{E}_1 + 3U_x,$$

where

$$U = u_1 \mathbf{E}_1 + u_2 \mathbf{E}_2 + \dots + u_n \mathbf{E}_n,$$

and

$$\mathbf{L} = T \mathbf{E}_1 + V(x) T^{-1} \mathbf{E}_1, \quad A = T^2 \mathbf{E}_1 + V(x+1) + V(x),$$

where

$$V(x) = v_1(x)\mathbf{E}_1 + v_2(x)\mathbf{E}_2 + \dots + v_n(x)\mathbf{E}_n.$$

Taking the tensor product of some Lie algebras, like in the above examples of pseudo-differential and shift operators, with the pseudo-scalar algebra  $ps(n)$  one can derive several examples of the coupled extensions of known integrable systems.

This is in fact the case of the coupled AKNS systems (10), where its zero-curvature equation,

$$\mathbf{L}_t - \mathbf{W}_x + [\mathbf{L}, \mathbf{W}] = 0, \quad (13)$$

is from the Lie algebra being tensor product of  $ps(2)$  with the loop algebra of  $sl(2)$ .

The respective generating operators are given by

$$\mathbf{L} = \mathbf{E}_1 \otimes U_1 + \mathbf{E}_2 \otimes U_2 = \begin{pmatrix} U_1 & U_2 \\ 0 & U_1 + U_2 \end{pmatrix}, \quad (14)$$

where

$$U_1 = \begin{pmatrix} -\lambda & p \\ q & \lambda \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix}$$

and

$$\mathbf{W} = \mathbf{E}_1 \otimes W_1 + \mathbf{E}_2 \otimes W_2 = \begin{pmatrix} W_1 & W_2 \\ 0 & W_1 + W_2 \end{pmatrix},$$

where

$$W_1 = \begin{pmatrix} -\lambda^2 + \frac{1}{2}pq & p\lambda - \frac{1}{2}p_x \\ q\lambda + \frac{1}{2}q_x & \lambda^2 - \frac{1}{2}pq \end{pmatrix}, \quad W_2 = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}$$

with

$$\begin{aligned} d &= \alpha \lambda^2 + \frac{1}{2}(1 - \alpha)(ps + qr + rs) - \frac{1}{2}\alpha pq, \\ e &= ((1 - \alpha)r - \alpha p)\lambda + \frac{1}{2}\alpha p_x - \frac{1}{2}(1 - \alpha)r_x, \\ f &= ((1 - \alpha)s - \alpha q)\lambda - \frac{1}{2}\alpha q_x + \frac{1}{2}(1 - \alpha)s_x. \end{aligned}$$

We have left here some freedom related to the parameter  $\alpha$  being integration constant from the computation of  $\mathbf{W}$ . For various details on derivations of integrable systems we send the reader to [11] and references therein. Then from the zero-curvature equation (13) we get the following system

$$\begin{aligned} p_t &= -\frac{1}{2}p_{xx} + p^2q, & q_t &= \frac{1}{2}q_{xx} - pq^2, \\ r_t &= \frac{1}{2}\alpha p_{xx} - \alpha p^2q + (1 - \alpha) \left( -\frac{1}{2}r_{2x} + 2pqr + 2prs + qr^2 + p^2s + r^2s \right) \\ s_t &= -\frac{1}{2}\alpha q_{xx} + \alpha pq^2 + (\alpha - 1) \left( -\frac{1}{2}s_{2x} + 2pqs + 2qrs + q^2r + ps^2 + rs^2 \right). \end{aligned} \quad (15)$$

The 2c-AKNS (10) (also obtained in [8]) one obtains for  $\alpha = 0$ . Whereas, the coupled AKNS obtained in [7] is generated for  $\alpha = -1$ . This system is in fact a linear composition of the symmetry (10) (the case of  $\alpha = 0$ ) and the one obtained from (15) for  $\alpha = 1$ .

In the papers [6]-[10] there are also other examples of integrable couplings for field and lattice soliton systems of the same class. All of them are generated by the spectral Lax operators, from the Lie algebra  $ps(2) \otimes sl(2)[[\lambda, \lambda^{-1}]]$ , being in the form (14), however with different entries of  $U_1$  and  $U_2$ .



## 6 Decoupling procedure

Consider the transformation of the pseudo-scalar algebra  $ps(n)$  in the form of a similarity relation

$$T(\mathbf{A}) := S^{-1} \mathbf{A} S,$$

where

$$S_{ij} = \begin{cases} 1 & \text{for } i \leq j \\ 0 & \text{for } i > j \end{cases}$$

and

$$(S^{-1})_{ij} = \begin{cases} 1 & \text{for } j = i \\ -1 & \text{for } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, this transformation is apparently an isomorphism of matrix algebras. One finds that

$$T(\mathbf{E}_k) = \begin{cases} 1 & \text{for } j = i, j \geq k \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for  $n = 4$

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and for  $\mathbf{A}$  given by (12)

$$T(\mathbf{A}) := \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 + a_2 & 0 & 0 \\ 0 & 0 & a_1 + a_2 + a_3 & 0 \\ 0 & 0 & 0 & a_1 + a_2 + a_3 + a_4 \end{pmatrix}.$$

The above transformation naturally extends to any tensor product of some algebra  $\mathfrak{g}$  with  $ps(n)$ , that is  $ps(n) \otimes \mathfrak{g}$ . Let

$$\mathbf{L} = \sum_{k=1}^n \mathbf{E}_k \otimes A_k, \tag{16}$$

where  $A_k$  belong to  $\mathfrak{g}$ . Then, the transformation extends to elements of the form (16) by the formula

$$T(\mathbf{L}) := \sum_{k=1}^n T(\mathbf{E}_k) \otimes A_k. \tag{17}$$

Since this transformation is an algebra isomorphism it preserves the Lie algebraic construction of the coupled systems. Combining (17) with the transformation of components:

$$\tilde{A}_k = \sum_{i=1}^k A_i \quad k = 1, \dots, n, \tag{18}$$

we decouple the whole construction into  $n$  independent copies, that is  $ps(n) \otimes \mathfrak{g}$  decouples into the direct product of  $n$  copies of  $\mathfrak{g}$ .

As consequence of the above decoupling procedure, the linear transformation to new field variables

$$\tilde{u}_k = \sum_{i=1}^k u_i \quad k = 1, \dots, n,$$

separates the coupled equations (4)-(5) to  $n$  copies of the basic equation (3) in new fields  $\tilde{u}_k$ , that is

$$(\tilde{u}_k)_t = K[\tilde{u}_k] \quad k = 1, \dots, n.$$

In particular this is the case of  $nc$ -KdV (6) and for  $nc$ -Volterra (7) systems.

In the case of the coupled AKNS system (10) or (15) generated by the Lax operator (14) the decoupling transformation (18) takes the form

$$\begin{aligned} \tilde{U}_1 &= U_1 & \tilde{p}_1 &= p & \tilde{q}_1 &= q \\ \tilde{U}_2 &= U_1 + U_2 & \tilde{p}_2 &= p + r \\ & & \tilde{q}_2 &= q + s, \end{aligned} \quad \Longleftrightarrow$$

where

$$\tilde{U}_i = \begin{pmatrix} -\lambda & \tilde{p}_i \\ \tilde{q}_i & \lambda \end{pmatrix} \quad i = 1, 2.$$

As result the coupled AKNS separates into two copies of the standard AKNS system.

All coupled triangular systems considered in previous sections as well as these from [6]-[10] (as they construction is based on the Lie algebra  $ps(2) \otimes sl(2)[[\lambda, \lambda^{-1}]]$  separate (decouple) into copies of the basic equations in the same manner. In fact the same situation is in the general case, when the underlying Lie algebra has the form  $ps(n) \otimes \mathfrak{g}$ . This follows from the fact that the above transformation is in the form of similarity relation which naturally preserves Lie commutator, trace form, Lax and zero-curvature representations, Hamiltonian structure and other related structures.

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